III Airy str $\rightsquigarrow$ spectral curve

$\sigma = (1, \dots, r)$

$x^{\alpha}(z) = \sum_{k \in \mathbb{Z}+\frac{1}{2}} \frac{J_k}{z^{k+1}}$ $\quad \alpha = 1, \dots, r$ (Hopper basis)

$J_k = \begin{cases} k\Phi_k & k>0 \\ 0 & k=0 \\ -k\Phi_{-k} & k<0 \end{cases}$ $\quad W^{\alpha}(x) = \mathrm{Res}_z \left(x^{\alpha}(z) \dots X(z) \right)$

$W^{\alpha}(z) = \frac{1}{r} \sum_{i=1}^{r} \sum_{k \in \mathbb{Z}+\frac{1}{2}} \frac{\alpha^{i}}{\zeta_i^{k-1}(-\zeta_i)^k} \sum_{k_1, \dots, k_r \in \mathbb{Z}} \left(\frac{\Phi_{k_1}^{(1)} \dots \Phi_{k_r}^{(r)}}{\zeta_1^{k_1} \dots \zeta_r^{k_r}} \right) \left(\frac{\zeta_i}{z} \right)^k$ $\quad \frac{\partial x}{\partial t} = + \frac{\partial z}{\partial z^2}$

$= \sum_{k \in \mathbb{Z}} \frac{W_k^{\alpha}}{z^{k+1}} = \prod_{\text{loop}} \sigma^{\alpha}(z)$

- $W_k^{\alpha} e^{\beta} = 0 \quad \forall (\alpha, k) \quad k \gg -\mathcal{O}(1)$ $\quad \left[k \geq -k_0 + (1-r)\mathcal{O} \right]$
  $r = \mathcal{O}(1)$ and $\mathcal{O}$
  Airy structure

$\rightsquigarrow W^{\alpha}(z) e^{\beta} = o(z^{-\mathcal{O}(1)})$

$z = z^{\sigma} = \alpha \, z^{-\mathcal{O}(1)} \quad \alpha = 0$

spinal bundle $\mathcal{B}$ = Spec $\mathbb{C}[[z]]$ $\quad$ ① z^{σ} $\quad$ $W^{\alpha}(z) e^{\beta}$ monomials with poles of countable order

base Spec $\mathbb{C}[[z]]$ $\quad$ ⊙ spinal multiplication

$\mathbb{C}$ spinal multiplication

$\mathbb{C}$-jet

$\mathbb{C}[[z]]$

zeros of dx $= \{\alpha, \dots, r\}$

α: zeros of order $r-1$

U_α: small neighborhood of α

From

- $F = \sum_{\substack{k_1, \dots, k_n \neq 0 \\ k_1 + \dots + k_n = 0}} \frac{\hbar^{g-1}}{n!} F_g[k_1, \dots, k_n] \, x_{k_1} \dots x_{k_n}$

  $x_k \to d\xi_{-k}$

  evaluate at in

  $\omega_{g,n}(z_1, \dots, z_n) = \sum_{k_1, \dots, k_n} F_g[k_1, \dots, k_n] \prod_{i=1}^{n} d\xi_{-k_i}(z_i)$

$\mathcal{G}_{\mathrm{Spec}}$

$\bigcup_{\alpha} U_\alpha$

$\downarrow$

$\mathbb{P}^1$

zeros of dx $= \{\alpha, \dots, r\}$

α: zeros of order $r-1$

U_α: small neighborhood of α

$\mathcal{G}_{\mathrm{Spec}} = \bigcup_{\alpha=1}^{r} U_\alpha$

$\downarrow$

$\mathbb{P}^1$

- $\mathcal{J}(z) := \sum_{k \in \mathbb{Z}} \frac{J_k}{z^{k+1}} dz$

  $= \underbrace{\mathcal{J}_+(z)}_{\substack{\text{positive} \\ \text{pieces of } z \\ \text{(regular modes)}}} + \underbrace{\mathcal{J}_-(z)}_{\substack{\text{negative} \\ \text{pieces of } z}}$

$H^{\ell}(x) = \mathcal{L} W^{\ell}(x) \mathcal{U}^{-1}$ $\qquad \mathcal{U} = \exp\left(\sum_{k>0} \frac{\overbrace{F_1[-k] J_k}^{\text{dilaton shift}}}{\hbar \cdot k} + \sum_{k,l>0} \frac{F_2[-k,-l]}{2kl\hbar} J_k J_l \right)$

exp in k, l

$\mathcal{J}(z)$ becomes $\quad \mathcal{J}_+(z) + \mathcal{J}_-(z) + \omega_{0,1}(z)$

where $\omega_{0,1}(z) = \sum_{k>0} F_1[-k] \, d\xi_k(z)$.

$d\xi_k(z) = z^{k-1} dz$

$\mathcal{J}_+(z) = \sum_{k>0} J_k \, d\xi_k(z)$

$\mathcal{J}_-(z) = \sum_{k>0} J_{-k} \, d\xi_{-k}(z)$.

$d\xi_{-k}(z) = \frac{dz}{z^{k+1}} + \sum_{l>0} \frac{F_2[-k,-l]}{k} d\xi_l(z)$

$= \mathrm{Res}_{z'=0} \left(\omega_{0,2}(\cdot, z) \right) \frac{dz}{z^{k+1}}$

★ $\omega_{0,2}(z_1, z_2) \overset{\text{def}}{=} \underbrace{\frac{dz_1 dz_2}{(z_1 - z_2)^2}}_{\omega_{0,2}^{\text{std}}} + \sum F_2[-k,-l] \, d\xi_k(z_1) d\xi_l(z_2)$

If $\mathcal{C}$ (blobbed) curve

$\omega_{0,2}$ a fundamental bidiff of second kind (double pole on diag, res of 1)

then ★ is the expansion $\substack{z_1 \to \alpha \\ z_2 \to \alpha}$ zeros of dx.

REFORMULATION OF W-CONSTRAINTS

- $z \in \mathcal{C}$ $\qquad f(z) = x^{\alpha}(x(z)) = \{ z, \Theta z, \Theta^2 z, \dots, \Theta^{r-1} z \}$

  $x = z^{\sigma}$ $\qquad \Theta^r = 1$ primitive

$\psi_{(g,n,\alpha)}^{(\alpha)} = \frac{1}{r!} \sum_{\alpha_1, \dots, \alpha_n = 0}^{r-1} \prod_{i=1}^{n} \frac{\Theta^{\alpha_i + 1 + i\alpha}}{(\Theta^{\alpha_i + i\alpha} - \Theta^{\alpha_i + i\alpha})^2} \prod_{i=1}^{n} \Theta^{-i\alpha_i(z_i)}$

we recognize $\omega_{0,2}^{\text{std}}(\Theta^{\alpha_i} z, \Theta^{\alpha_j} z)$.

$H_i(x) = \sum_{\substack{Z \subset f(z) \\ |Z| = i}} \frac{\hbar^g}{2^g g! (-2)^i} \prod_{l \in Z} \omega_{0,2}(z_{2l-1}, z_{2l}) \prod_{l \in A_0} \omega_{0,1}(z_l)$

$\sum_{\substack{Z \subset f(z) \\ |Z| = i \\ 0 \leq g \in \mathbb{Z}/2 \\ A_0 \cup A_{+1} \cup A_- = f(z) \setminus Z}}$

$\prod_{l \in A_+} \overline{\mathcal{J}}_+(z_l) \prod_{l \in A_-} \overline{\mathcal{J}}_-(z_l)$

$= \sum_{\substack{Z \subset f(z) \\ |Z| = i}} \sum_{\substack{U \cup A_0 \cup A_{+1} \cup A_- = Z \\ P \in \mathrm{Short}(f(z))}} \prod_{\{l, l'\} \in P} \omega_{0,2}(z_l, z_{l'}) \prod_{l \in A_0} \omega_{0,1}(z_l)$

$\prod_{l \in A_+} \underbrace{\overline{\mathcal{J}}_+(z_l)}_{\text{pieces with diff op}} \prod_{l \in A_-} \underbrace{\overline{\mathcal{J}}_-(z_l)}_{\text{pieces with diff op}}$

when acting on $e^{\mathcal{F}}$:

$x_k \longrightarrow d\xi_{-k}(z)$ on terms involving x_k, x_l

called $\frac{\partial}{\partial t^k}$ $\quad k = 1, \dots, n$.

$\boxed{r = 2}$ $\quad \sum_{\substack{l \leq 2 \\ \sigma \in f(z)}} \left[\sum_{\substack{\sigma', \sigma'' \in f(z) \\ \sigma' \sigma'' = \sigma}} \omega_{g-1,n+2}(z_\sigma', z_\sigma'', \underline{z}_S) + \sum_{\substack{g_1 + g_2 = g \\ J \cup J' = S}} \omega_{g_1,n+1}(z_\sigma', \underline{z}_J) \, \omega_{g_2,n+1}(z_\sigma'', \underline{z}_{J'}) \right]$

$= o(z^{-2\mathcal{O}(1)})$ $\qquad$ this contains the $\omega_{0,n+2}$

higher r $\quad \sum_{\substack{z \in f(z) \\ |z| = i}} \omega + \omega \cdot \omega + \omega \cdot \omega \cdot \omega + \dots\dots = o(z^{-r\mathcal{O}(1)})$ $\qquad$ Abstract loop equations

Thm $\quad$ If these equations have a solution, it must be given by the (spectral curve) topological recursion $\qquad$ (exactness $\Leftrightarrow$ Airy structure)

$r = 2$ $\quad$ Chekhov-Eynard-Orantin

$\omega_{g,n}(z, z_2, \dots, z_n) = \mathrm{Res}_{z'=0} \left(\frac{\int_0^{z'} \omega_{0,2}(\cdot, z_2)}{(\omega_{0,1}(z) - \omega_{0,1}(z'))} \right) \left[\omega_{g-1,n+1}(z, z', z_2, \dots, z_n) + \sum_{\substack{g_1 + g_2 = g \\ I \cup J = \{2, \dots, n\}}} \omega \cdot \omega \right]$

$f(z) = \{z, -z\}$ $\qquad$ (recursion).

$r > 2$ $\quad$ Bouchard-Eynard recursion.

Idea of proof

- $\sum_{\alpha \in \mathbb{Z}} H_\alpha u^{\alpha - 2} = \mathcal{U} \sum \mathbb{C}[\cdot] u^{\cdot} \mathcal{U}$

  specialize to $u = -\omega_{0,1}(z) \sim \star \, z^{s}$

  $s = \min \{ m > 0 \quad F_m[-m] \neq 0 \}$

  $H_i e^{\mathcal{F}} = o(z^{-r\mathcal{O}(1)})$ $\qquad (\omega_{0,1}(z))^{-i} = O(z^{(i-1)s})$
  $|z| = 1$

  $\rightsquigarrow$ combination $\sum_i H_i (-\omega_{0,1}(z))^{-i} = O(z^{-r\mathcal{O}(1)})$

  $\sum \left(\omega_{g,n+2} \prod_{\substack{\sigma \in f(z) \\ \neq \{\sigma\}}} (\omega_{0,1}(z) - \omega_{0,1}(z')) \right) = O(z^{-r\mathcal{O}(1)})$ $\qquad$ contribution from disjoint

  $+ \boxed{\omega^2 \text{ involving only } \omega_{g,n} \quad 2g - 2 + n \leq 2g - 2 + n}$

- $\omega_{g,n+1}(z, z_2, \dots, z_n) = \mathrm{Res}_{z'=0} \left(\int_0^{z'} \omega_{0,2}(\cdot, z) \right) \boxed{\omega_{g,n+1}(z', z_2, \dots, z_n)}$

  Cauchy residue formula (by definition of $d\xi_{-k}$).
  the $O(z^{-\cdot})$ doesn't contribute (compensates by zeroes in $\omega_{0,1}$)

$\begin{cases} x = z^{\sigma} \\ y \, \omega_{0,1} = z^{1-\sigma} dz \end{cases}$ $\qquad$ Written r-spin

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Other automorphisms $\quad$ (Bouchard, Hadad, 20. B, Kramer, Schaller).

- $\sigma = (\underbrace{1 \dots r_1}) (r_1+1, \dots, r_2) \dots (r_{k-1}+1, \dots, r_k)$

  $\sum r_i = r$

- $(x_{\mu,k})$ $\quad \mu = 1, \dots, d$ $\quad$ d copies of Heisenberg
  $k > 0$

  dilation $\quad x_{\mu,k} \longrightarrow x_{\mu,k} - \frac{\ell_{\mu}}{\sqrt{k}}$

Thm $\quad$ (BKS 20)

$d \geq 2$ $\quad s_\mu \in \{1, r_\mu + 1\}$, order μ set $\quad \frac{s_1}{r_1} \leq \dots \leq \frac{s_d}{r_d}$.

$r_\mu = -1 \mod s_{\mu+1}$, $\quad s_{\mu+1} = \dots = s_{\mu+n} = 1$, $\quad r_\mu = 1 \mod s_\mu$.

$\ell_\mu^{r_\mu} \neq \ell_\nu^{r_\nu}$ $\quad$ whenever $r_\mu s_\nu = r_\nu s_\mu$ $\quad \forall \mu, \nu$

Then we get an Airy structure

Spectral curve

$x = z^{r}$

$\omega_0 = y \, dx \Rightarrow y = z^{s-r}$

$d = 1$ $\qquad x^{r-s} y^{r} = 1$

$\left[\prod_{\mu} \left(x^{r_\mu - s_\mu} \left(\frac{y}{\ell_\mu} \right)^{s_\mu} - 1 \right) = 0 \right]$

singular curve $\mathcal{C}$

$\left[\hat{\mathcal{C}} \text{ normalization} : \begin{array}{l} \text{dim of unique sol} \\ x^{r-s} y^{r} = \prod_{\mu} (\ell_\mu^{-1})^{s_\mu} \end{array} \right]$

We can define TR on those singular curves

Φ_1, Φ_2,

$\mathcal{U} = \exp\left(\sum_{\substack{2g-2+n>0 \\ k_1, \dots, k_n > 0}} \frac{\overbrace{\Phi_{g,n}[k_1, \dots, k_n]}^{\text{blobbed}}}{k_1 \dots k_n} \hbar^{g-1} J_{k_1} \dots J_{k_n} \right)$

preserves Airy str $\qquad \omega_{g,n} \overset{\Delta}{=} \mathrm{Res}_{z=0} \left(\left(\omega_{0,2} \right) + \widetilde{\mathcal{F}_{g,n}} \right)$

blobbed TR : general solutions
(B, Shadrin 15) $\qquad$ Abstract loop